§ Hyperbolic restriction on $\mathcal{U}_G^d$

Choose $\rho : \mathbb{C}^* \to T$ 1PS

$\rightsquigarrow$ $G \leftarrow P \twoheadrightarrow L$ group

$$\mathcal{U}_G^d \xleftarrow{j} \mathcal{U}_P^d \xrightarrow{p} \mathcal{U}_L^d \qquad \text{Uhlenbeck}$$

Rem. $\uparrow$ $\nearrow$ depend only on L and P respectively

$$\Phi = \Phi_{L,G}^{P} := p_* j^! : D^b_T(\mathcal{U}_G^d) \longrightarrow D^b_T(\mathcal{U}_L^d)$$

∘ associativity

$G \supset P \supset Q$ parabolics $\quad Q_L =$ image of Q in L

$$\begin{array}{ccc} G \supset & P \supset & Q \\ & \downarrow & \downarrow \\ & L \supset & Q_L \\ & & \downarrow \\ & & M \end{array}$$

$$\Longrightarrow \quad \Phi_{M,G}^{Q} = \Phi_{M,L}^{Q_L} \circ \Phi_{L,G}^{P}$$

$$\because \quad \mathcal{U}_Q^d \cong \mathcal{U}_P^d \times_{\mathcal{U}_L} \mathcal{U}_{Q_L}^d \qquad //$$

We fix B and consider only parabolics compatible with B hereafter

∘ Recall

$$\mathcal{U}_G^d = \bigsqcup \underbrace{\mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2}_{= \mathrm{Bun}_{G,\lambda}^{d'}} \qquad \lambda = (1^{m_1} 2^{m_2} 3^{m_3} \cdots)$$

$\mathrm{Stab}\,\lambda = S_{m_1} \times S_{m_2} \times \cdots$

χ: irrep. of $S_{m_1} \times S_{m_2} \times \cdots$

$\longrightarrow$ local system on $S_\lambda \mathbb{A}^2$

$\mathrm{Perv}(\mathcal{U}_G^d) =$ additive subcategory of perverse sheaves which are direct sums of

$$IC(\mathrm{Bun}_{G,\lambda}^d, \chi)$$

$\underline{\mathrm{Th}}$ $\Phi_{L,G} : \mathrm{Perv}(\mathcal{U}_G^d) \longrightarrow \mathrm{Perv}(\mathcal{U}_L^d)$

enough to estimate dimension for the fiber :

- the most basic (largest) sheaf $IC(\mathcal{U}_G^d)$
- two basic stratum
  a) Bun_L^d
  b) the most singular stratum $S_{(d)}\mathbb{A}^2$

— a) Bun_L^d (no singularity)

$$\mathrm{Bun}_G^d \overset{j}{\hookleftarrow} \underset{\substack{\| \\ \{\text{framed } P\text{-b'dles}\}}}{\mathrm{Bun}_P^d} \xrightarrow{p} \mathrm{Bun}_L^d$$

"extensions" in generalized sense

$\mathcal{F}_1 \in \mathrm{Bun}_G^d$ $\quad T_{\mathcal{F}_1}\mathrm{Bun}_G^d = H^1(\mathbb{P}^2, \mathfrak{g}_{\mathcal{F}_1}(-l_\infty))$

$\mathfrak{g}_{\mathcal{F}_1} = \mathcal{F}_1 \times_{Ad} \mathfrak{g}$

$\mathcal{F} \in \mathrm{Bun}_L^d$ $\quad$ Note $\mathfrak{g} = \underbrace{\mathfrak{l} \oplus \mathfrak{n}^+}_{\mathfrak{p}} \oplus \mathfrak{n}^-$

$$\rightsquigarrow T_{\mathcal{F}_1}\mathrm{Bun}_G^d = \underset{\substack{\| \\ T_{\mathcal{F}_1}\mathrm{Bun}_L^d}}{H^1(\mathfrak{l}_{\mathcal{F}_1}(-l_\infty))} \oplus \underset{\substack{\| \\ \mathrm{Leaf}_\alpha}}{H^1(\mathfrak{n}^+_{\mathcal{F}_1}(-l_\infty))} \oplus \underset{\substack{\| \\ \mathrm{Leaf}_\alpha^-}}{H^1(\mathfrak{n}^-_{\mathcal{F}_1}(-l_\infty))}$$

$\therefore$ p : vector b'dle $\Rightarrow$ $p_* j^! \mathbb{C}_{\mathrm{Bun}_G^d} \cong \mathbb{C}_{\mathrm{Bun}_L^d}$
rank $= \frac{1}{2}$ codim

Thom isom. !

$$\mathrm{Hom}(IC(\mathrm{Bun}_L^d), p_* j^! IC(\mathcal{U}_G^d)) \cong \mathbb{C} \times (\text{Thom isom.})$$

(canonically trivialized 1 dim space)
$\longrightarrow$ (Thom isom.) will be omitted

∘ the most singular stratum $\mathcal{C}_{S(d)\mathbb{A}^2}$

$$U^d \equiv U^{d,P}_{L,G} := \mathrm{Hom}(\mathcal{C}_{S(d)\mathbb{A}^2}, \Phi^P_{L,G}(IC(\mathcal{U}^d_G)))$$

$$\rightsquigarrow \quad U^d_{L,G} \otimes \mathcal{C}_{S(d)\mathbb{A}^2} \longrightarrow \underbrace{\Phi^P_{L,G}(IC(\mathcal{U}^d_G))}_{\text{isotypic component}}$$

basic stratum + factorization $\longrightarrow$

$$\Phi^P_{L,G}(IC(\mathcal{U}^d_G)) \cong \bigoplus_{\lambda, d'} IC(\mathrm{Bun}^{d'}_L \times S_\lambda \mathbb{A}^2, (U^1)^{\otimes m_1} \otimes (U^2)^{\otimes m_2} \otimes \cdots)$$

$\lambda = (1^{m_1} 2^{m_2} \cdots)$

representation of $S_{m_1} \times S_{m_2} \times \cdots$

Take cohom $H^*_{\mathbb{T}}$ $\longrightarrow$ nontrivial local system do not contribute

$$\bigoplus_d H^*_{\mathbb{T}}(\Phi^P_{L,G}(IC(\mathcal{U}^d_G)) \cong \bigoplus_d IH^*_{\mathbb{T}}(\mathcal{U}^d_L) \otimes \mathrm{Sym}(U^1 \oplus U^2 \oplus \cdots)$$

<u>Fact</u> [BF] $\dim U^d = \mathrm{rank}\, G - \mathrm{rank}[L, L]$ (indep. of d)

e.g. $L = T \Rightarrow \mathrm{rank}\, G$

$\therefore$ $\mathrm{gdim}\, \mathrm{Sym}(U^1 \oplus U^2 \oplus \cdots) = \mathrm{gdim}$ Fock for $\mathrm{Heis}.(\mathfrak{h})$

$\mathfrak{h} = \mathrm{Lie}\, T$

<u>1st goal</u>: Construct a "natural" isomorphism

$$U^d \cong \mathfrak{h}$$

s.t. $$H^*_{\mathbb{T}}(\Phi^B_{T,G}(IC(\mathcal{U}^d_G))) \cong \text{Fock Heis}(\mathfrak{h})$$

$$= \mathrm{Sym}_{H^*_{\mathbb{T}}(pt)}(z^{-1}\mathfrak{h}[z^{-1}])$$

∘ On dimension estimates

⟷ More concrete description of $\mho^d$

$$\mathcal{U}^d_G = {}^c\mathcal{U}^d_G \times \mathbb{A}^2 \qquad \text{etc}$$

$\uparrow$ centered G-bdles

$$\mho^d = \mathrm{Hom}\big(\mathcal{C}_{d,0}, \overset{= p_* j^!}{\Phi_{L,G}}(IC({}^c\mathcal{U}^d_G))\big)$$

$$\cong H^0\big(\underset{\underset{\mathcal{U}^d_{P,0}}{\|}}{p^{-1}(d,0)}, \tilde{j}^!\, IC({}^c\mathcal{U}^d_G)\big)$$

$$\begin{tikzcd}
{}^c\mathcal{U}^d_G & {}^c\mathcal{U}^d_P \arrow[l, "j"'] \arrow[r, "p"] & {}^c\mathcal{U}^d_L \\
 & \cup & \cup \\
 & p^{-1}(d,0) \arrow[uul, "\tilde{j}"] \arrow[r] & \{(d,0)\}
\end{tikzcd}$$

Prop $\dim \mathcal{U}^d_{P,0} \leq d h^\vee - 1$

(all irr. components have $\leq$)

or $\dim \mathcal{U}^d_{P,0} \cap \mathrm{Bun}^d_G \leq d h^\vee - 1$

Prop $\mho^d = H_{[0]}(\mathcal{U}^d_{P,0} \cap \mathrm{Bun}^d_G)$

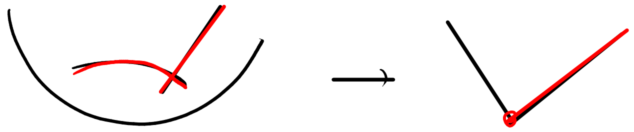
$\uparrow$ spanned by irreducible components of $\dim = d h^\vee - 1$

In particular, it has a base.

Ex. $G = SL_2 \supset B$

$$\begin{array}{ccc} \mathcal{U}_G^d & \hookleftarrow & p^{-1}(d\cdot 0) \\ \uparrow & & \uparrow \text{ proper} \\ \tilde{\mathcal{U}}_2^d & \longleftarrow & \star \end{array}$$

$\star = \{0 \to I_1 \to E \to I_2 \to 0\}$



$I_1 \in \pi^{-1}(d_1 0)$, $\dim = d_1 - 1$ or 0

$I_2 \in \pi^{-1}(d_2 \cdot 0)$, $d_2 - 1$ or 0

$$\dim \mathrm{Ext}^1(I_1, I_2(-l_\infty)) = d_1 + d_2$$

$$\therefore\ d_1 + d_2 + d_1 - 1 + d_2 - 1 = 2d - 2 < dh^\vee - 1$$

$\therefore$ Only $d_1 = d$ or $d_2 = d$ is possible

Lemma If $I_2 = \mathbb{C}[x,y]$ $(d_2 = 0)$ $\Rightarrow E \in \pi^{-1}(d \cdot 0)$ ($2d-1$ dim)

$\therefore\ p^{-1}(d\cdot 0) = \pi$ ($d_1 = 0$ case)

(e.g. $0 \to \mathcal{O} \to \mathcal{O}^{\oplus 2} \to \mathcal{O} \to \mathbb{C}_0 \to 0$ Koszul res.

$\longrightarrow 0 \to \mathcal{O} \to \mathcal{O}^{\oplus 2} \to m_0 \to 0$ $\to d = 1$ case

Rem. This example shows

$\mathcal{U}_P^d \cap \mathrm{Bun}_G^d \neq \mathrm{Bun}_P^d$ In general

($\mathcal{U}_P^d \cap \mathrm{Bun}_G^d$: "quasi" P-bdles) associativity

$$\Phi^{B}_{T,G} \cong \Phi^{B}_{T,L_i} \circ \Phi^{P_i}_{L_i,G} \qquad (i \in I)$$

$$\Phi^{B}_{T,G}(IC(\mathcal{U}^d_G)) \cong \Phi^{B_{L_i}}_{T,L_i} \circ \underbrace{\Phi^{P_i}_{L_i,G}(IC(\mathcal{U}^d_G))}_{\cong\ IC(\mathcal{U}^d_{L_i}) \oplus (U^d_{L_i,G} \otimes \mathcal{C}_{S(d)\mathbb{A}^2}) \oplus \text{ other summands not contribute to } U^d}$$

$U^{d,B}_{T,G} = \mathrm{Hom} \ni \mathcal{C}_{S(d)\mathbb{A}^2}$

$* \ \Phi^{P_{L_i}}_{T,L_i}(\mathcal{C}_{S(d)\mathbb{A}^2}) = \mathcal{C}_{S(d)\mathbb{A}^2}$

$$\therefore \underbrace{U^d_{T,G}}_{\dim = \operatorname{rank} G} \cong \underbrace{U^d_{T,L_i}}_{1\text{-dim.}} \oplus U^d_{L_i,G}$$

$\Rightarrow \mathrm{Sym}(\bigoplus_d U^d_{T,G}) \cong \mathrm{Sym}(\oplus U^d_{T,L_i}) \otimes \mathrm{Sym}(U^d_{L_i,G})$

Similar argument $\Rightarrow$
$G = SL_r$

$$\underbrace{\mathrm{Hom}(\mathcal{C}_{S(d)\mathbb{A}^2}, \Phi_{T,G}(\pi_* \mathcal{C}_{\tilde{\mathcal{U}}^d_r}))}_{\cong\ \mathfrak{h}_{gl_r}} \cong \underbrace{U^d_{T,G}}_{\cong\ \mathfrak{h}_{sl_r}} \oplus \underbrace{\mathbb{C}[\pi^{-1}(d,0)]}_{\cong\ \mathbb{C} \times (\text{scalar})}$$

From these, we can now arrive at the 1st goal:

Note $[L_i, L_i] \cong SL_2 \quad \therefore \mathrm{Sym}(\oplus U^d_{T,L_i})$ has a Heis. action.

from the [MO] for $r=2$ i.e., (P^-_m)

$\rightsquigarrow$ Set P^i_m for each i

$\rightsquigarrow [P^i_m, P^j_n] = -m\delta_{m,n} \frac{1}{\varepsilon_1\varepsilon_2}(\alpha_i, \alpha_j)$

from SL_3 & $SL_2 \cdot SL_2$ reduction

§ W-algebra

Th. $\bigoplus_d IH^*_{\mathbb{T}}(\mathcal{U}^d_G)\otimes\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$ has a "natural"

structure of the Verma module of $W_k(\mathfrak{g})$

dual Coxeter # $\quad$ let $h^\vee = -\frac{\varepsilon_2}{\varepsilon_1}$, highest wt $\lambda = \frac{\vec{a}}{\varepsilon_1} - \rho$

$\mathrm{Lie}\,\mathbb{T} = \langle \varepsilon_1, \varepsilon_2, a^1, \dots, a^l\rangle$

NB W-symmetry
$\vec{a}$: usual action
λ : dot action

$i^! j^! \longrightarrow i^* j^!$

$$\bigoplus_d IH^*_{\mathbb{T},c}(\mathcal{U}^d_G) \xrightarrow{☆} \bigoplus_d H^*_{\mathbb{T},c}(\mathcal{U}^d_T, \Phi_{T,G}(\mathcal{U}^d_G)) \longrightarrow$$

$\| \qquad\qquad\qquad\qquad \|$

$M_A(\vec{a}) \qquad\qquad N_A(\vec{a})$

$$\left(\longrightarrow \bigoplus_d H^*_{\mathbb{T}}(\mathcal{U}^d_T, \Phi_{T,G}(\mathcal{U}^d_G)) \longrightarrow \bigoplus_d H^*_{\mathbb{T}}(\mathcal{U}^d_G)\right)$$

$\qquad\qquad \| \qquad\qquad\qquad\qquad$

$\qquad DN_A(-\vec{a}) \qquad\qquad DM_A(-\vec{a})$

- free over $H^*_{\mathbb{T}}(pt)$
- all isom. $\otimes\,\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$

$\implies$ injective

☆ factors through

$$\begin{tikzcd} IH^*_{\mathbb{T},c}(\mathcal{U}^d_G) \arrow[r, hook] \arrow[dr, hook] & H^*_{\mathbb{T},c}(\Phi_{T,G}(\mathcal{U}^d_G)) \\ & H^*_{\mathbb{T},c}(\mathcal{U}^d_{L_i}, \Phi_{L_i,G}(\mathcal{U}^d_G)) \arrow[u, hook] \end{tikzcd}$$

$$\therefore\ IH^*_{\mathbb{T},c}(\mathcal{U}^d_G) \underset{☆☆}{\subset} \bigcap_i H^*_{\mathbb{T},c}(\mathcal{U}^d_{L_i}, \Phi_{L_i,G}(\mathcal{U}^d_G))$$

$\subset H^*_{\mathbb{T},c}(\Phi_{T,G}(\mathcal{U}^d_G))$

$\subset IH^*_{\mathbb{T}}(\mathcal{U}^d_G)$

$\otimes\,\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$

Lemma ☆☆ is $=$

($\longrightarrow$ used for the integrable form)

[Feigin–Frenkel]

Suppose $k \left(= -\frac{\varepsilon_2}{\varepsilon_1} - h^\vee\right)$ is generic

$$\Longrightarrow \quad W_k(\mathfrak{g}) \cong \bigcap_i \underline{\mathrm{Vir}_i \otimes \mathrm{Heis}(\alpha_i^\perp)} \subset \mathrm{Heis}(\mathfrak{h})$$

(in the sense of the vertex algebra)

Kernel of S_i : screening

<u>NB.</u> central charge of the Virasoro $1 + \frac{6(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}$

$\varepsilon_1 \leftrightarrow \varepsilon_2$

$$\mathrm{Vir}(k) \cong \mathrm{Vir}(k') \quad \text{if } (k+2)(k'+2) = 1$$

$$\Longrightarrow \quad W_k(\mathfrak{g}) \cong W_{k'}(\mathfrak{g}) \quad \text{if } (k+h^\vee)(k'+h^\vee) = 1$$

(for ADE)

Now get $\quad W_k(\mathfrak{g}) \curvearrowright M_{\mathbb{A}}(\vec{a}) \otimes \mathbb{C}(\mathrm{Lie}\,\mathbb{T})$

highest wt $\longleftarrow$ formula of 0-mode

§ Integral form

W-algebra defined over $\mathbb{C}[\varepsilon_1,\varepsilon_2]$

can set $\varepsilon_1=0 \longrightarrow k=\infty$

$\varepsilon_2=0 \longrightarrow k=-h^\vee$ critical level

$\longrightarrow$ commutative

Recall $[P_m(\alpha), P_n(\beta)] = -\langle\alpha,\beta\rangle m\,\delta_{m+n,0}\,\mathrm{id}$

$IH^*_{T,c}(\mathcal{U}^d_G)$ $\Rightarrow \alpha = \mathrm{P.D.}[0] \in H^2_{(\mathbb{C}^*)^2,c}(\mathbb{A}^2) = \varepsilon_1\varepsilon_2$

cpt support

$\therefore$ $\widetilde{P}^i_n := P^i_n(\varepsilon_1\varepsilon_2)$

$[\widetilde{P}^i_m, \widetilde{P}^j_n] = -\varepsilon_1\varepsilon_2(\alpha_i,\alpha_j)\,m\,\delta_{m+n,0}\,\mathrm{id}$

$\widetilde{L}^i_n = [G(\psi), \widetilde{P}^{+}_n \text{ for } L^i] = L^i_n \times \varepsilon_1\varepsilon_2$

$\Longrightarrow [\widetilde{L}^i_m, \widetilde{L}^i_n] = \varepsilon_1\varepsilon_2\left\{(m-n)\widetilde{L}^i_{m+n} + \left(\varepsilon_1\varepsilon_2 + 6(\varepsilon_1+\varepsilon_2)^2\right)\delta_{m,-n}\frac{m^3-m}{12}\right\}$

<u>Define</u> $W_A(\mathfrak{g})$ = cohomology of the BRST cpx $/A$

$A = \mathbb{C}[\varepsilon_1,\varepsilon_2]$

<u>Th</u> $W_A(\mathfrak{g}) = \bigcap_i \mathrm{Vir}_A \otimes \mathrm{Heis}_A(\alpha_i^\perp)$

$\mathrm{Vir}_A = \langle \widetilde{L}^i_m \rangle$, $\mathrm{Heis}_A(\alpha_i^\perp) = \langle \text{proj. of } \widetilde{P}^j_m \rangle$

<u>Rem</u>. In usual approaches, the second parameter is hidden in the natural filtration on $W_k(\mathfrak{g})$ (Rees alg. constr.)

Cor. (of Lemma before) $M_{\mathbb{A}}(\vec{a})$ is a $W_{\mathbb{A}}(\mathfrak{g})$-module
universal Verma module

($M_{\mathbb{A}}(\vec{a})$ is cyclic under $W_{\mathbb{A}}(\mathfrak{g})$)

$N_{\mathbb{A}}(\vec{a})$ is naturally a Heisenberg module
universal Wakimoto module

$$M_{\mathbb{A}}(\vec{a}) \otimes_{H^*_{\mathbb{T}}(pt)} \mathbb{C}$$
$H^*_{\mathbb{T}}(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2, \vec{a}]$

Set $\varepsilon_1, \varepsilon_2, \vec{a}$ to scalars.

$\Rightarrow$ any Verma module is obtained by a specialization.

Two applications of integral forms
1) Whittaker vector
2) character formula (in preparation)

Q. $M_{\mathbb{A}}(\vec{a}) \otimes_{\mathbb{A}_T} \mathbb{C}$ is not irreducible for special highest wts
(e.g. minimal model)
How to understand this
compute irreducibles ?

$\longrightarrow$ A. Analysis of fixed pts.
& recover Arakawa's result

$\longrightarrow$ Watch Braverman's talk in summer.

§ Whittaker property

$\ell = \operatorname{rank} \mathfrak{g}$

$d_1 = 1 \leqq d_2 \leqq \cdots \leqq d_\ell = h^\vee - 1$: exponents of $\mathfrak{g}$

$d_k + 1$: degree of generators of $S(\mathfrak{g})^G$

e.g. $d_1 = 1, d_2 = 2, \cdots, d_\ell = \ell$ for $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$

NB. d_k is multiplicity free, except $d_{\ell/2} = d_{\ell/2+1} = \ell - 1$ for D_ℓ: even

$$F^{(k)} \in S(\mathfrak{h})^W = S(\mathfrak{g})^W \quad (k = 1, \cdots, \ell) \quad \text{generators}$$
$$\parallel$$
$$F^{(k)}(h^1, \cdots, h^\ell) \qquad h^i : \text{simple coroot}$$

So $S(\mathfrak{h})^W = \mathbb{C}[F^{(1)}, \cdots, F^{(\ell)}] \leftarrow$ its W-alg is chiralization

<u>Rem</u> $F^{(k)}$ is unique up to constant multiple except case

But topology should give a natural Integral structure.

So should well-defined (at least), up to ± 1.

<u>Fact</u>. $W_A(\mathfrak{g})$ has "generating" fields $\widetilde{W}^{(k)} = \sum_{n \in \mathbb{Z}} \widetilde{W}^{(k)}_n z^{-n-d_k-1}$

s.t. $F^{(k)}\left(\sum_{n<0} \widetilde{P}^{(i)}_n z^{-n-1} \right) = \sum_{n<0} \widetilde{W}^{(k)}_n z^{-n-d_k-1}$ at $\varepsilon_1 = \varepsilon_2 = 0$

$$W_A(\mathfrak{g})\big|_{\varepsilon_1 = \varepsilon_2 = 0} \cong \mathbb{C}[\widetilde{W}^{(k)}_n \mid n < 0, \ k = 1, 2, \cdots]$$

e.g. <u>$k=1$</u> (Virasoro) $\widetilde{L}_n = -\frac{1}{4} \sum_{n<\ell<0} \widetilde{P}_\ell \widetilde{P}_{n-\ell}$

<u>Conjecture</u> (Gaiotto for SL_2 $\longrightarrow$ obvious to generalize)

Let $|1^d\rangle \in IH^*_{\mathbb{T}}(\mathcal{U}^d_G)$ fundamental class

$$\Longrightarrow \quad \widetilde{W}^{(k)}_n |1^d\rangle \underset{\star}{=} \begin{cases} |1^{d-1}\rangle & k = \ell, \ n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\langle\ |\ \rangle$: Kac-Shapovalov form (— intersection pairing)

$$\widetilde{W}[\vec{\lambda}] = \widetilde{W}^{(1)}_{-\lambda^1_1}\widetilde{W}^{(1)}_{-\lambda^1_2}\cdots\ \widetilde{W}^{(\ell)}_{-\lambda^\ell_1}\widetilde{W}^{(\ell)}_{-\lambda^\ell_2}\cdots \qquad \vec{\lambda} = (\lambda^1, \lambda^2, \cdots, \lambda^\ell)$$

ℓ-tuple of partitions

$\Rightarrow \{\widetilde{W}[\vec{\lambda}]\}$ base of $M_{\mathbb{A}}(\vec{a})$

$$\langle -\vec{a} | \theta(\widetilde{W}[\vec{\lambda}])\widetilde{W}[\vec{\mu}] | \vec{a}\rangle = K_{\vec{\lambda}\vec{\mu}} \quad : \text{KS matrix}$$

$$☆ \Rightarrow \quad |1^d\rangle = \sum K^{\vec{\mu}\vec{\lambda}_0}\ \widetilde{W}[\vec{\mu}]|\mathrm{vac}\rangle$$

where $\vec{\lambda}_0 = (\phi, \cdots, \phi, (1^d))$

$K^{\vec{\mu}\vec{\lambda}_0}$ is, in principle, computable.

Th Conjecture is true up to sign.

(Sketch of the proof)

$\widetilde{W}^{(k)}$ has degree $d_k + 1$

$$\widetilde{W}^{(k)}_n : |1^d\rangle \in IH^{[-2d h^\vee]}_{\mathbb{T}}(\mathcal{U}^d_G) \longrightarrow IH^{[2(d_k+1)-2dh^\vee]}_{\mathbb{T}}(\mathcal{U}^{d-n}_G)$$

$$IH^{[-2d h^\vee]}_{\mathbb{T}}(\mathcal{U}^d_G) = IH^0_{\mathbb{T}}(\mathcal{U}^d_G), \qquad IH^{[2(d_k+1)-2dh^\vee]}_{\mathbb{T}}(\mathcal{U}^{d-n}_G) = IH^{2(d_k+1-nh^\vee)}_{\mathbb{T}}(\mathcal{U}^{d-n}_G)$$

$$0 \leq d_k + 1 - n\underbrace{h^\vee}_{= d_\ell + 1} \iff k = \ell,\ n = 1$$

$$\therefore \widetilde{W}^{(k)}_n |1^d\rangle = \begin{cases} \exists c_d |1^{d-1}\rangle,\ c_d \in \mathbb{C} & \text{if } k = \ell, n = 1 \\ 0 & \text{otherwise} \end{cases}$$

We prove $c_d^2 = c_1^{2d} \quad \forall d$.

Since $\widetilde{W}^{(\ell)}$ is well-defined only up to scalar, we can set $c_1 = 1$, so we are done.

To prove $c_d^2 = c_1^2$.

We study $\langle 1^d | 1^d \rangle$ and $\langle w^d | w^d \rangle$ at $\varepsilon_1 = \varepsilon_2 = 0$,

$|w^d\rangle = \sum K^{\vec{\mu}\vec{\lambda}_0} \widetilde{W}[\vec{\mu}]|vac\rangle$? $K^{\vec{\lambda}_0\vec{\lambda}_0}$

More precisely $(\varepsilon_1\varepsilon_2)^d \langle 1^d | 1^d \rangle \Big|_{\varepsilon_1, \varepsilon_2 = 0} = \frac{1}{d!}\left(\varepsilon_1\varepsilon_2 \langle 1^d | 1^d \rangle \Big|_{\varepsilon_1 = \varepsilon_2 = 0} \right)^d$

and same is true for w^d

- geometric side

$\langle 1^d | 1^d \rangle$ (Nekrasov partition function)

$$\frac{1}{d!\,(\varepsilon_1\varepsilon_2)^d} = \int_{S^d\mathbb{A}^2} 1$$

$$\zeta_*: H_*^T(S^d\mathbb{A}^2) \xrightarrow{\otimes \mathbb{C}(\mathrm{Lie}\,T)} H_*^T(\mathcal{U}_G^d)$$

$$\zeta_*^{-1}[\mathcal{U}_G^d] = d!\,(\varepsilon_1\varepsilon_2)^d \langle 1^d | 1^d \rangle \Big|_{\varepsilon_1 = \varepsilon_2 = 0} \times [S^d\mathbb{A}^2]$$

$\Longrightarrow$ Use factorization

- algebraic side

Consider $M_{\mathbb{A}}(\vec{a}) \otimes_{\mathbb{A}_T} \underbrace{\mathbb{R}_T}_{\text{regular fct at } \varepsilon_1 = \varepsilon_2 = 0} \Big|_{\varepsilon_1 = \varepsilon_2 = 0}$

and divide it by radical of $(\varepsilon_1\varepsilon_2)^d \langle 1 \rangle \Big|_{\varepsilon_1 = \varepsilon_2 = 0}$

$\Longrightarrow$ all higher Fourier mode vanish

and compute Poisson bracket for $\{\widetilde{W}_{-1}, \widetilde{W}_1\}$ at $\varepsilon_1 = \varepsilon_2 = 0$.